Semi-Generalized Closed Mappings and Generalized-Semi Closed Mappings and Its Relationships with Semi-Normal and Semi-Regular Spaces

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Abstract
In this paper, the semi-generalized closed maps (sg-closed maps) and generalized semi-closed maps (gs-closed maps) are studied and some results are presented and proved including the study of some of their basic properties which are related to such type of mappings. Also, a study of s-normal spaces and s-regular spaces is given using its relationships with sg-closed and gs-closed mappings.

1- Introduction
P. Bahtharyya and P. K. Lahiri [2] introduced the concept of semi-generalized closed sets (sg-closed sets) and investigated some of their properties and semi-T_{1/2} spaces.
S. P. Arya and T. Nour [1] defined the generalized semi-open sets (gs-open sets) and studied some of their properties and characterizations of s-normal spaces by using semi-open sets.
Recently, P. Sundaram introduced the concept of semi-generalized continuous map and generalized semi-continuous.

In this paper, the study of sg and gs-closed maps and some of their basic properties are given, then its relationship with s-normal and s-regular spaces are also presented as the main results.

2- Preliminaries
In this section, the basic definitions and concepts related to this paper are given for completeness:

Definition (2.1), [3]:
A subset of a topological space \((X, \tau)\) is said to be semi-open set, if there exists an open set \(U \subseteq X\) such that \(U \subseteq S \subseteq \text{cl}(U)\), where \(\text{cl}\) refers to the closure.
It is remarkable that the complement of semi-open set is said to be semi-closed set. The semi-closure of subset \(A\) of \((X, \tau)\) denoted by \(\text{scl}(A)\) or briefly \(\text{scl}(A)\), is defined to be the intersection of all semi-closed sets containing \(A\).

Definition (2.2), [2]:
A subset \(A\) of \((X, \tau)\) is said to be semi-generalized closed (written in short as sg-closed) in \((X, \tau)\) if \(\text{scl}(A) \subseteq O\), whenever \(A \subseteq O\) and \(O\) is semi-open in \((X, \tau)\).

Also, a subset \(B\) is said to be semi-generalized open (written as sg-open) in \((X, \tau)\) if its complement \(X-B\) is sg-closed in \((X, \tau)\).
\(A \subseteq X\) is called sg-closed in \(X\) if and only if for all \(U\) semi-open set in \(X\), \(A \subseteq U \implies \text{scl}(A) \subseteq U\).

Following are some of the basic well known results and remarks concerning gs-closed sets:
1. The complement of sg-closed is called sg-open.
2. A set \(A\) is sg-closed if and only if \(\text{scl}(A) - A\) contains no non-empty semi-closed.
3. Let \(A\) be sg-closed, then \(A\) is semi-closed if and only if \(\text{scl}(A) - A\) is semi-closed.
4. If \(A\) is sg-closed and \(A \subseteq B \subseteq \text{scl}(A)\), then \(B\) is sg-closed.
5. Every s-closed set is sg-closed, but the converse is not true.
6. g-closed and sg-closed are in general independent and every semi-closed set is sg-closed.

Now, we consider the other type of topological sets concerning this paper.

Definition (2.3), [1]:
A subset \(A\) of \((X, \tau)\) is said to be generalized semi-open (written as gs-open) in \((X, \tau)\) if \(F \subseteq \text{sint}(A)\), whenever \(F \subseteq A\) and \(F\) is closed in \((X, \tau)\). A subset \(B\) is generalized semi-closed (written as gs-closed) if its complement \(X-B\) is gs-open in \((X, \tau)\).

The following results appeared in [1] and [2] which are given here for completeness:

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Theorem (2.4): A subset \( A \) of \( (X, \tau) \) is gs-closed in \( X \) if and only if \( \text{scl}(A) \subset U \), whenever \( A \subset U \) and \( U \) is open in \( (X, \tau) \).

Proposition (2.5): If \( A \) is an open and gs-closed set of \( (X, \tau) \), then \( A \) is semi-closed set.

Proposition (2.6): Let \( F \subseteq A \subseteq X \), where \( A \) is an open set in \( X \) and also gs-closed in \( X \). If \( F \) is gs-closed set in \( A \), then \( F \) is gs-closed set in \( X \).

Proposition (2.7): Let \( F \subseteq A \subseteq X \), where \( A \) is an open in \( X \) and if \( F \) is gs-closed set in \( X \), then \( F \) is gs-closed set in \( A \).

3- Semi-Generalized And Generalized Semi-Closed Maps
In this section, we will give the definition of semi-generalized closed maps and generalized semi-closed maps and some related results.

Definition (3.1), [7]: A map \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be semi-closed if for any closed set \( F \) of \( X \), \( f(F) \) is semi-closed in \( Y \).

Definition (3.2), [6]: A map \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be g-closed if for any closed set of \( X \), \( f(F) \) is g-closed in \( Y \).

Definition (3.3), [8]: A map \( f : X \rightarrow Y \) is called a generalized semi-closed map (written as gs-closed map) if for each closed set \( F \subset X \), \( f(F) \) is gs-closed set of \( Y \).

Next, we give some of the obtained results concerning the mappings of this section:

Theorem (3.4): A map \( f : (X, \tau) \rightarrow (Y, \sigma) \) is sg-closed if and only if for each subset \( S \) of \( Y \) and for each open set \( U \) containing \( f^{-1}(S) \), there is a sg-open set \( V \) of \( Y \) such that \( S \subseteq V \) and \( f^{-1}(V) \subseteq U \).

Proof: (Necessity) Let \( S \) be a subset of \( Y \) and \( U \) be an open set of \( X \), such that \( f^{-1}(S) \subset U \).
Then \( Y - f(X - U) \), say \( V \), is a sg-open set containing \( S \), such that \( f^{-1}(V) \subseteq U \).

(Sufficiency) Let \( F \) be a closed set of \( X \), then \( f^{-1}(Y - f(F)) \subset X - F \) and \( X - F \) is open.
By hypothesis, there is a sg-open set \( V \) of \( Y \) such that \( Y - f(F) \subset V \) and \( f^{-1}(V) \subset X - F \).
Therefore, we have \( F \subset X - f^{-1}(V) \) and hence:
\[ Y - V \subset f(F) \subset f(X - f^{-1}(V)) \subset Y - V. \]
Which implies \( f(F) = Y - V \), since \( Y - V \) is sg-closed.
\( f(F) \) is sg-closed and thus \( f \) is a sg-closed map.

Theorem (3.5): A mapping \( f : (X, \tau) \rightarrow (Y, \sigma) \) is gs-closed if and only if for each subset \( S \) of \( Y \) and for each open set \( U \) containing \( f^{-1}(S) \), there exists a gs-open set \( V \) of \( Y \) containing \( S \) and \( f^{-1}(V) \subset U \).

Proof: (Necessity) Let \( S \) be a subset of \( Y \) and \( U \) be an open set of \( X \), such that \( f^{-1}(S) \subset U \).
Then \( Y - f(X - U) \), say \( V \), is a gs-open set containing \( S \) such that \( f^{-1}(V) \subset U \).
(Sufficiency) Let \( F \) be a closed set of \( X \), we claim that \( f(F) \) is gs-closed in \( Y \), that is, \( f^{-1}(Y - f(F)) \subset X - F \).
By taking \( S = Y - f(F) \) and \( U = X - F \) hypothesis there exists a gs-open set \( V \) of \( Y \) containing \( Y - f(F) \) and \( f^{-1}(V) \subset X - F \).
Then we have \( F \subset X - f^{-1}(V) \) and \( Y - V = f(F) \).
Since \( Y - V \) is gs-closed, \( f(F) \) is gs-closed and thus \( f \) is a gs-closed map.

The main results of this paper are given in the next section:

4- S-Normal And S-Regular Spaces
In this section, we study s-normal and s-regular spaces and we give sufficient conditions on \( f : (X, \tau) \rightarrow (Y, \sigma) \) so that \( f \) preserved s-normality and s-regularity.

First, recall the following definitions:

Definition (4.1), [5]: Let \( (X, \tau) \) be a topological space, then \( X \) is s-normal if and only if given a closed set \( F \subseteq X \) and \( x \notin F \), then there exists two disjoint open sets \( W_1 \) and \( W_2 \) such that \( x \in W_1 \), \( F \subseteq W_2 \) and \( W_1 \cap W_2 = \emptyset \).

Definition (4.2), [5]: Let \( (X, \tau) \) be a topological space, we say that \( (X, \tau) \) is s-normal if and only if given two disjoint closed sets, \( F_1 \) and \( F_2 \) in \( X \), then there
exists two disjoint semi-open sets, $W_1$ and $W_2$ such that $F_1 \subseteq W_1$, $F_2 \subseteq W_2$.

**Theorem (4.3):**

If $f : (X, \tau) \longrightarrow (Y, \sigma)$ is a continuous and onto gs-closed map from a normal space $(X, \tau)$ to a space $(Y, \sigma)$, then $(Y, \sigma)$ is s-normal.

**Proof:**

Let $A$ and $B$ be disjoint closed sets of $Y$. Since $X$ is normal, then there exist disjoint open sets $U$ and $V$ in $X$ such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$ (by theorem (3.5)).

Then there exist gs-open sets $G$ and $H$ in $Y$ such that $f^{-1}(G) \subseteq U$, $f^{-1}(H) \subseteq V$ and $f^{-1}(G) \cap f^{-1}(H) = \emptyset$.

Hence $G \cap H = \emptyset$, since $G$ is gs-open and $A$ is closed.

$G \supseteq A$ implies that $s \text{ int}(G) \supseteq A$.

Similarly, $s \text{ int}(H) \supseteq B$.

Hence $s \text{ int}(G) \cap s \text{ int}(H) = G \cap H = \emptyset$.

Therefore, $Y$ is s-normal.

The next corollary is given in [7], which may be considered as a result of the above theorem:

**Corollary (4.5):**

(i) Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a continuous semi-closed onto mapping, if $(X, \tau)$ is normal then $(Y, \sigma)$ is s-normal.

(ii) Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a continuous, sg-closed onto mapping, if $(X, \tau)$ is normal then $(Y, \sigma)$ is s-normal.

**Proof:**

(i) Since $f$ is semi-closed and then $f$ is gs-closed.

Then by theorem (4.3), we get that $Y$ is s-normal.

(ii) Since $f$ is sg-closed, then $f$ is gs-closed.

Then by theorem (4.3), we get that $Y$ is s-normal.

**Theorem (4.6):**

If $f : (X, \tau) \longrightarrow (Y, \sigma)$ is a continuous semi-open and gs-closed onto mapping from a regular space $(X, \tau)$ to a space $(Y, \sigma)$, then $(Y, \sigma)$ is s-regular.

**Proof:**

Let $y \in Y$, let $U$ be an open set containing $y$ in $Y$.

$f$ is onto, then there exists $x \in X$ such that $f(x) = y$.

Now, $f^{-1}(U)$ is an open set in $X$ containing $x$.

But $X$ is regular, then there exist an open set $V$ such that:

$x \in V \subseteq \text{cl}(U) \subseteq f^{-1}(U)$

$y \in f(V) \subseteq f(\text{cl}(V)) \subseteq U$.

But $f(\text{cl}(V))$ is gs-closed.

Then we have $\text{scl}(f(\text{cl}(V))) \subseteq U$.

Therefore, $Y \in f(V) \subseteq \text{scl}(f(V)) \subseteq U$ and $f(V)$ is semi-open in $Y$ (Because $f$ is semi-open).

Hence $Y$ is s-regular.

**Corollary (4.7):**

If $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a continuous, semi-open and sg-closed onto mapping, if $(X, \tau)$ is a regular space then $(Y, \sigma)$ is s-regular.

**Proof:**

Since $f$ is sg-closed, then $f$ is gs-closed (by theorem (4.6)).

Hence we get that $Y$ is s-regular.

5- References


الخلاصة

في هذا البحث، قمنا بدراسة التطبيقات شبه العامة المغلقة (sg-closed maps) والتطبيقات العامة شبه المغلقة (gs-closed maps) وvatتغطي بعض النتائج حولهما مع البرهان حيث تضمنت دراسة بعض من خواصهما الرئيسة الملاحظة لهذا النوع من التطبيقات. كما وتمت دراسة الفضاءات شبه الطبيعية (s-normal) والفضاءات شبه المنتظمة (s-regular) بالاعتماد على علاقتها مع التطبيقات شبه العامة المغلقة والتطبيقات العامة شبه المغلقة.